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NAVAL POSTGRADUATE SCHOOL

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SECOND ORDER NECESSARY CONDITIONS FOR

PROBLEMS WITH STATE INEQUALITY CONSTRAINTS

by

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ABSTRACT:

A set of second order necessary conditions is obtained for solution to a basic problem in optimal control involving the state constraints

$$\psi^{\alpha}(t,x) \leq 0 \qquad \alpha = 1,\ldots,m.$$

For the case of a normal solution arc, the methods of Hestenes are used to show that the second variation of a certain functional is non-negative on a class of variations of the solution arc. The results for this problem are basic in the sense that they can be extended to include analagous results for much more complicated problems.

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1. Introduction:

A general problem in optimal control can be stated as follows:

Let C be the class of arcs

$$a: \qquad x^i(t) \qquad u^k(t) \qquad b^\sigma \qquad t^o \le t \le t^1$$

$$i = 1,\ldots,N \qquad k = 1,\ldots,k \qquad \sigma = 1,\ldots,r$$

whose elements $(t,x(t),u(t),b)$ lie in a region R in txub space and which in addition have $u(t)$ piecewise continuous and satisfy the constraints

$$\dot{x}^i = f^i(t,x,u,b) \qquad i = 1,\ldots,N$$

$$\psi^\alpha(t,x,b) \le 0 \qquad \alpha = 1,\ldots,m' \qquad \psi^\alpha(t,x,b) = 0 \qquad \alpha = m'+1,\ldots,m$$

$$\theta^\eta(t,x,u,b) \le 0 \qquad \eta = 1,\ldots,L' \qquad \theta^\eta(t,x,u,b) = 0 \qquad \eta = L'+1,\ldots,L$$

$$I_\gamma(a) \le 0 \qquad \gamma = 1,\ldots,p' \qquad I_\gamma(a) = 0 \qquad \gamma = p'+1,\ldots,p$$

$$x^i(t^s) = X^{is}(b) \qquad i = 1,\ldots,N \qquad s = 0,1$$

where $I_\gamma(a) = g_\gamma(b) + \int_{t^o}^{t^1} L_\gamma(t,x(t),u(t),b)dt \qquad \gamma = 1,\ldots,p$

It is desired to minimize the integral

$$I_o(a) = g_o(b) + \int_{t^o}^{t^1} L_o(t,x(t),u(t),b)dt$$

on the class C.

With the dimensions of our terms allowed to assume new values, the above problem is expressible as one in which the constraints have the simpler basic form shown below and also an assumption made on the rank of the matrix $(\psi^{\alpha}_{x} f_{u})$ in the general problem is transformed into a stronger condition in the basic problem.

The basic constraints are:

$$\dot{x}^{i} = f^{i}(t,x,u) \qquad i = 1,\ldots,N.$$

$$\psi^{\alpha}(t,x) \leq 0 \qquad \alpha = 1,\ldots,m$$

$$I_{\gamma}(a) \leq 0 \qquad \gamma = 1,\ldots,p' \qquad I_{\gamma}(a) = 0 \qquad \gamma = p' + 1,\ldots,p$$

$$x^{i}(t^{s}) = X^{is}(b) \qquad s = 0,1 \qquad i = 1,\ldots,N$$

The methods of Hestenes are used to show that when a solution arc to the basic problem is normal, then the second variation of a certain functional is non-negative on a class of variations of that solution arc. Extension of this result to the general problem is the subject of a succeeding paper.

2. A Basic Problem

In what follows we shall assume familiarity with [1] and [2] and shall use quantities defined there. In addition, unless otherwise specified, the conventions and notations of those papers will also apply here.

A slight generalization of the problem of Section 4 of [1] appears in the following statement.

Let C be the class of arcs

$$a: \qquad x^i(t) \qquad u^k(t) \qquad b^\sigma \qquad t^o \le t \le t^1$$

$$i = 1,\ldots,N, \qquad k = 1,\ldots,K, \qquad \sigma = 1,\ldots,r$$

whose elements $(t,x(t),u(t))$ and b lie respectively in open sets R in txu space and B in b space and which in addition have $u(t)$ piecewise continuous. The terms x^i are called state variables. The terms u^k, b^σ are called control variables and control parameters respectively. We require these arcs to satisfy the constraints.

(1a) $$\dot{x}^i = f^i(t,x,u) \qquad i = 1,\ldots,N$$

(1b) $$\psi^\alpha(t,x) \le 0 \qquad \alpha = 1,\ldots,m$$

(1c) $$I_\gamma(a) \le 0 \quad (\gamma = 1,\ldots,p') \qquad I_\gamma(a) = 0 \quad \gamma = p'+1,\ldots,p$$

(1d) $$x^i(t^s) = X^{is}(b) \qquad s = 0,1$$

where:

$$I_\gamma(a) = g_\gamma(b) + \int_{t^o}^{t^1} L_\gamma(t,x(t),u(t))dt \qquad \gamma = 1,\ldots,p$$

(1) Numbers in brackets refer to references

(2) Unless otherwise noted the indices i, k, σ, α will have the respective ranges $i = 1,\ldots,N$, $k = 1,\ldots,K$ $\sigma = 1,\ldots,r$, $\alpha = 1,\ldots,m$

It is desired to minimize the integral

$$I_o(a) = g_o(b) + \int_{t^o}^{t^1} L_o(t,x(t),u(t))dt \tag{2}$$

on the class C

We suppose that the arc

$$a_o: \qquad x_o(t) \qquad u_o(t) \qquad b_o \qquad t^o \le t \le t^1$$

is a solution to our problem. In addition we set $\phi^\alpha = \psi^\alpha_t + \psi^\alpha_{x^i} f^i$ [(1)] and except for the functions $X^{js}(b)$ $j = 1,\ldots,N$ $s = 0,1$ which are assumed to be C^1 on B we assume all other functions to have the continuity properties introduced in Section 4 of [1]. The set R_o [(2)] is defined as the set of points (t,x,u) in R satisfying

$$\psi^\alpha \le 0 \tag{3a}$$

$$\phi^\alpha \ge 0 \quad \forall\alpha \ni \psi^\alpha = 0 \quad \text{or} \quad \phi^\alpha \le 0 \quad \forall\alpha \ni \psi^\alpha = 0 \qquad \alpha = 1,\ldots,m \tag{3b}$$

Finally let $\bar{D}$ be the set of points $(t,x_o(t),u)$ in R_o with $u = u_o(t)$ or for arbitrary u, with t interior to an interval of continuity of $u_o(t)$. Then we assume that the matrix

$$(\phi^\alpha_{u^k}) \qquad \alpha = 1,\ldots,m \tag{4}$$

has rank m on $\bar{D}$

In stating the above problem, the primary modification we have made to the problem of Section 4 of [1] was in allowing the functions ψ^α to have arbitrary form.

(1) For a function $M(t,x,u,b)$, the notations M_{x^i}, M_{u^k}, M_{b^σ}, M_t denote first partial derivatives with respect to the indicated variable. Also unless otherwise specified, repeated indices will be summed.

(2) We shall at a later point strengthen these assumptions for proving the second order theorems to follow.

An examination of the proof of Theorem 6.1 of [1] will show that with the following minor modifications the statements of that theorem apply to the problem introduced above: the symbols $N+m$, $K+m$, $r+2m$ are replaced by N,K,r respectively and the function $G(b)$ has the form

$$(5) \quad G(b) = \lambda_o g_o + \lambda_\gamma g_\gamma + \lambda_{p+i} X^{io} \qquad \gamma = 1,\ldots,p \qquad i = 1,\ldots,N$$

We require two properties of the multipliers $\mu_\alpha(t)$ not proven in Theorem 6.1 of [1]. The first of these properties is established in the following lemma.

Lemma 2.1. The multipliers $\mu_\alpha(t)$ are constant on intervals upon which $\psi^\alpha(t) < 0$.

Proof: The proof of this is a simple modification of a proof in [1]. Fix $\alpha = \bar{\alpha}$, $1 \le \bar{\alpha} \le m$ and let $[t',t'']$ be an interval such that $\psi^{\bar{\alpha}}(t) < 0$ there. Referring to Section 13 of [1], let w be an arc in the class W satisfying

$$w^\Gamma(t) = 0 \qquad \Gamma = 1,\ldots,K \qquad \Gamma \neq \bar{\alpha} \qquad t^o \le t \le t^1$$

$$w^{\bar{\alpha}}(t) = 0 \qquad t\varepsilon[t^o, t'] \cup [t'',t^1].$$

Referring to the argument involving (169) and (170) of [1] we see that those relations hold now also so that corresponding to (171) of [1] we obtain

$$(6) \qquad \int_{t'}^{t''} -\mu_{\bar{\alpha}} \dot{w}^{\bar{\alpha}} dt \ge 0$$

where according to the definition of the class W, the sign of $w^{\bar{\alpha}}$ is arbitrary on $[t',t'']$. Then by the Fundamental Lemma in the Calculus of Variations, the multiplier $\mu_{\bar{\alpha}}(t)$ is constant on $[t',t'']$.

While the construction of the proof has been with $[t',t'']$ interior to $[t^o,t^1]$, nevertheless, continuity properties of $\mu_\alpha(t)$ establish it also for all subintervals $[t',t'']$ of $[t^o,t^1]$, thus proving the lemma.

The second property is that

$$\mu_\alpha(t^1) = 0 \quad \text{if} \quad \psi^\alpha(t^1) < 0 \qquad \alpha = 1,\ldots,m$$

This property is established by an argument directly analagous to that used in [3] on page 372 and will not be repeated here.

3. Normality

Let $q_{\gamma j}(t)$ and $z_{ij}(t)$ $(\gamma = 0,1,\ldots,p, \quad i,j = 1,\ldots,N)$ be the functions of (66) in [1] and as in (70) of [1] define the functionals

(7a) $J_\gamma(a) = q_{\gamma j}(t^o)[x^j(t^o) - X^{jo}(b)] + I_\gamma(a) \qquad \gamma = 0,1,\ldots,p$

(7b) $J_{p+i}(a) = z_{ij}(t^o)[X^{jo}(b) - x^j(t^o)] + x^i(t^1) - X^{i1}(b)$

(7c) $J_{p+N+i}(a) = X^{io}(b) - x^i(t^o) \qquad i,j = 1,\ldots,N$

Next, denote by δa, the triple of vectors

$$\delta a: \qquad \delta x^i(t) \qquad \delta u^k(t) \qquad \delta b^\sigma \qquad t^o \le t \le t^1$$

(with $\delta x(t)$, $\delta u(t)$ respectively continuous and piecewise continuous functions.)

We call δa a variation of the arc a_o and for any function $M(t,x,u,b)$ which possesses first partial derivatives with respect to x,u,b along a_o,[1] we call

$$\delta M(t) = M_{x^i}(t)\delta x^i(t) + M_{u^k}(t)\delta u^k(t) + M_{b^\sigma}(t)\delta b^\sigma$$

the variation of M along a_o due to the variation δa.

We call a variation δa admissible[2] if it satisfies

$$\delta \dot{x}^i = \delta f^i \qquad i = 1,\ldots,N$$

$$\delta \psi^\alpha(t) \le 0 \quad \text{on a neighborhood of } s^\alpha \qquad \alpha = 1,\ldots,m$$

where $s^\alpha \equiv \{t \mid \psi^\alpha(t,x_o(t)) = 0\}$

Next, define the expressions

(8a) $J'_\gamma(a_o,\delta a) = q_{\gamma j}(t^o)[\delta x^j(t^o) - \delta X^{jo}] + I'_\gamma(a_o,\delta a)$

(8b) $J'_{p+i}(a_o,\delta a) = z_{ij}(t^o)[\delta X^{jo} - \delta x^j(t^o)] + \delta x^i(t^1) - \delta X^{i1}$

(1) Unless otherwise noted, for a function $Z(t,x,u,b)$ the notation $Z(t)$ will denote $Z(t, x_o(t), u_o(t), b_o)$ for the arc a_o under consideration.

(2) See section 11 of [1].

(8c) $J'_{p+N+i}(a_o,\delta a) = \delta X^{io} - \delta x^i(t^o)$

as variations of the functionals (7) due to the variation δa of a_o and where

$$I'_\gamma(a_o,\delta a) \equiv \delta g_\gamma + \int_{t^o}^{t^1} \delta L_\gamma \, dt \qquad \gamma = 0,1,\ldots,p$$

Now let γ_k be those indices for which

(9) $$I_\gamma(a_o) < 0 \qquad 1 \le \gamma \le p'$$

and let $\bar{p}$ be the number of such indices. According to Theorem 6.1 of [1], then with λ_γ as the multipliers of that theorem we have

$$\lambda_{\gamma_k} = 0 \qquad k = 1,\ldots,\bar{p}$$

Next let $\mu_\alpha(t)$ be the functions of Theorem 6.1 of [1] and define the solution arc a_o to be a normal solution if there are $\Omega = p-\bar{p}+2N$ (where $\bar{p}$ is the constant introduced above) admissible variations $\delta_\omega a$ $(\omega=1,\ldots,\Omega)$ of a_o which satisfy the conditions

(10a) $$\delta_\omega \psi^\alpha(t^o) = 0 \qquad \text{if } \mu_\alpha(t^o) \neq 0$$

(10b) $$\delta_\omega \psi^\alpha(t) = 0 \text{ on a neighborhood of } s^\alpha \qquad \alpha = 1,\ldots,m$$

in addition to providing that the matrix

(10c) $$(J'_\rho(a_o,\delta_\omega a)) \qquad \rho = 1,\ldots,p+2N, \quad \rho \neq \gamma_k, \quad \omega = 1,\ldots,\Omega$$

is non singular.

Define the class Y of admissible variations

$$Y: \qquad y^i(t) \qquad v^k(t) \qquad z^\sigma \qquad t^o \le t \le t^1$$

of a_o which in addition satisfy[1]

(1) The notation $\delta_Y \psi^\alpha(t)$ denotes the variation in the functions $\psi^\alpha(t)$ due to Y. Similar notation will also be used for other functions.

(11a) $\delta_y \psi^\alpha(t^c) = 0$ if $\mu_\alpha(t^c) \neq 0$ $c = 0,1$

(11b) $\delta_y \psi^\alpha(\bar{t}) = 0$ if $\mu_\alpha(t)$ is not constant on a neighborhood of $\bar{t}$

(11c) $\delta_y \psi^\alpha(t^o) = 0$ if $\psi^\alpha(t^o) = 0$ $1 \leq \alpha \leq m$

(11d) $J'_\gamma(a_o, y) = 0$ if $\lambda_\gamma \neq 0$ $1 \leq \gamma \leq p'$

(11e) $J'_\gamma(a_o, y) \leq 0$ if $\lambda_\gamma = 0$ $\gamma \neq \gamma_k$ $1 \leq \gamma \leq p'$

(11f) $J'_\rho(a_o, y) = 0$ $p' < \rho \leq p+2N$

where γ_k are the indices of (9) and where the multipliers λ_γ, $\mu_\alpha(t)$ are those of Theorem 6.1 of [1].

At this point we include in our assumptions the following conditions: i) the functions ψ^α are of class c^3 and c^1 respectively with respect to x and t and ψ^α_t is of class c^2 with respect to x while the functions f^i, L_γ, g_γ, X^{ic} are of class c^2; ii) the function $u_o(t)$ is continuous on $[t^o, t^1]$. For this problem we will prove the following theorem.

Theorem 3.1 Let a_o be a normal solution to our problem. Then the multipliers λ_ρ, $\mu_\alpha(t)$, $p_i(t), k^\alpha$ $\rho = 0,1,\ldots,p+N$ of Theorem 6.1 of [1] satisfy: i) $\lambda_o \neq 0$, and ii) with $\lambda_o = 1$, these multipliers are unique. Furthermore with G and H as the terms of Theorem 6.1 of [1] then for each variation y in the class Y, we have that

$$(12)\quad 0 \leq \Big[\big(p_i(t^c)X^{ic}_{b^\sigma}\big)_{c=0}^{c=1} + G_{b^\sigma b^\tau} + K^\alpha \psi^\alpha_{x^i x^j}(t^o)X^{io}_{b^\sigma} X^{jo}_{b^\tau}\Big] z^\sigma z^\tau$$

$$- \int_{t^o}^{t^1} \Big[H_{x^i x^j} y^i y^j + 2H_{x^i u^k} y^i v^k + H_{u^h u^k} v^h v^k\Big] dt$$

$$i,j = 1,\ldots,N \qquad h,k = 1,\ldots,K \qquad \sigma,\tau = 1,\ldots,r \qquad \alpha = 1,\ldots,m$$

4. An Imbedding Lemma

In order to prove this result we will first prove the following lemma with the aid of the proof of Lemma 11.1 of [1].

Lemma 4.1. Let $a(\beta)$ be the $\bar{h}$ parameter family of arcs

$a(\beta)$: $\quad x(t,\beta) \quad u(t,\beta) \quad b(\beta) \quad |\beta| \leq \beta^o, \quad h = 1,\ldots,\bar{h}$

associated with the admissible variations

$\delta_h a$: $\quad \delta_h x(t) \quad \delta_h u(t) \quad \delta_h b \quad h = 1,\ldots,\bar{h}$

as constructed in Lemma 11.1 of [1]. Then

$$(13a) \quad \frac{\partial x^i}{\partial \beta_h}(t,0) = \delta_h x^i(t)$$

$$(13b) \quad \frac{\partial u^k}{\partial \beta_h}(t,0) = \delta_h u^k(t)$$

$$(13c) \quad \frac{\partial b^\sigma(0)}{\partial \beta_h} = \delta_h b^\sigma \qquad h = 1,\ldots,\bar{h} \qquad t^o \leq t \leq t^1$$

In addition, the functions

$$(14a) \quad \frac{\partial^2 x^i(t,\beta)}{\partial \beta_h \partial \beta_s}$$

$$(14b) \quad \frac{\partial^2 u^k(t,\beta)}{\partial \beta_h \partial \beta_s}$$

$$(14c) \quad \frac{\partial^2 b^\sigma(\beta)}{\partial \beta_h \partial \beta_s} \qquad h,s = 1,\ldots,\bar{h} \qquad t' \leq t \leq t'' \qquad |\beta| \leq \beta^o$$

exist and are continuous where: i) β^o is the constant referred to above and ii) $[t',t'']$ is an interval of continuity of $\delta_h u(t)$ $(1 \le h \le \bar{h})$.

Proof: As a first step, we note according to the continuity properties of the functions ψ^α and their derivatives together with the definition of the functions ϕ^{α} (1), that these latter functions together with their mixed partial derivatives up through the second order with respect to x and u are continuous on R. Hence these statements hold also for the functions $\phi^{m+1},\ldots,\phi^k$ introduced in Section 8 of [1]. Referring to Lemma 10.1 of [1] and using these additional continuity properties of ϕ^Γ we see that the functions $U(t,x,s)$ of that lemma are continuous and have continuous first and second order mixed partial derivatives with respect to x, and s on a neighborhood of the arc a_o.

Furthermore, by referring to the remarks following (121) of [1], the existence and continuity of the functions

$$(15)\quad \frac{\partial x^i(t,\beta)}{\partial \beta_h} \qquad \frac{\partial u^k(t,\beta)}{\partial \beta_h} \qquad |\beta| \le \beta^o \qquad [t' \le t \le t'']$$

(where $u(t,\beta)$ is defined as $U(t,x(t,\beta), s(t,\beta))$ as in [1]) is established, with $[t',t'']$ as an interval of continuity of $\delta_h u$ $(1 \le h \le \bar{h})$.

Now define the $N\bar{h}$ functions

$$(16)\quad K_h^j(t,k,\beta) = f^j_{x^i} k^i_h + f^j_{u^\tau}\left[U^\tau_{x^i} k^i_h + U^\tau_{s^\Gamma}\frac{\partial s^\Gamma(t,\beta)}{\partial \beta_h}\right] \qquad \begin{array}{l} i,j = 1,\ldots,N \\ h = 1,\ldots,\bar{h} \\ \tau,\Gamma = 1,\ldots,K \end{array}$$

(1) See remarks below 2)

where i) the functions U are those of lemma 10.1 of [1] and ii) the arguments of the derivatives of f^j and U^τ are $(t, x(t,\beta), s(t,\beta))$ as described in (115), (118) and (119-1) of [1].

By the properties of the functions f, $x(t,\beta)$, U, $s(t,\beta)$, we may solve the system of differential equations (1)

$$(17a)\quad \dot{k}_h^i = K_h^i$$

$$\dot{\bar{\beta}} = 0$$

with

$$(17b)\quad k_h^i(t^o,\beta) = \frac{\partial x^i(t^o,\beta)}{\partial \beta_h}$$

$$\bar{\beta}(t^o) = \beta \qquad h = 1,\ldots,\bar{h}$$

for $|\beta| \leq \beta^o$, where this is the constant referred to above. We thus obtain the family of functions

$$(18)\quad k_h^i(t,\beta) \qquad h = 1,\ldots,\bar{h} \qquad t^\delta \leq t \leq t^1 \qquad |\beta| \leq \beta^o.$$

Now according to the definition of $x^i(t^o,\beta)$ given in (119-2) of [1] together with Lemma 10.4 of [1] we see that the functions $\frac{\partial x^i}{\partial \beta_h}(t^o,\beta)$ are of class C^1. Then by the theorem referred to above, we see that the functions $k_h^j(t,\beta)$ and $\frac{\partial k_h^j(t,\beta)}{\partial \beta_s}$ $h,s = 1,\ldots,\bar{h}$ are continuous on intervals $[t',t'']$ upon which $\delta_1 u(t), \ldots, \delta_{\bar{h}} u(t)$ are continuous.

Next, by referring to the functions K_h^j of (16), we see that the

(1) See for example Theorem 14.2 of chapter 1 of reference [3]. Also we have included the parameter β as the variable $\bar{\beta}$ in this system.

functions $\frac{\partial x^j}{\partial \beta_h}(t,\beta)$ of (15) also satisfy the system (17). Then by the uniqueness of solutions to (17) we obtain

$$k_h^i(t,\beta) = \frac{\partial x^i}{\partial \beta_h}(t,\beta) \qquad h = 1,\ldots,\bar{h} \tag{19}$$

Furthermore by the properties of the functions $k_h^j(t,\beta)$ already established, we see that

$$\frac{\partial^2 x^i}{\partial \beta_h \partial \beta_s}(t,\beta) \qquad h,s = 1,\ldots,\bar{h} \qquad |\beta| \le \beta^o \qquad t' \le t \le t'' \tag{20a}$$

also exist and are continuous, where $[t',t'']$ is an interval upon which $\delta_1 u(t),\ldots, \delta_{\bar{h}} u(t)$ are continuous.

By (20a) together with the definition of $s(t,\beta)$ and the properties of the functions U of Lemma 10.1 of [1] as described in the remarks above (15), we see that the functions

$$\frac{\partial^2 u^k(t,\beta)}{\partial \beta_h \partial \beta_s} \qquad h,s = 1,\ldots,\bar{h} \qquad |\beta| \le \beta^o \qquad [t',t''] \tag{20b}$$

are also continuous(where $[t',t'']$ is as described below (20a)).

Finally, by using the functions $\theta^o(\beta)$ of (121) of [1], we see that the above statement also holds for

$$\frac{\partial^2 b(\beta)}{\partial \beta_h \partial \beta_s} \qquad h,s = 1,\ldots,\bar{h} \qquad |\beta| \le \beta^o \qquad [t',t''] \tag{20c}$$

Thus statement (14) is proven.

In order now to establish statement (13), we observe that (119-2) and (114) of [1] yield

$$\frac{\partial x^i(t^o,0)}{\partial \beta_h} = \delta_h x^i(t^o) \qquad h = 1,\ldots,\bar{h} \tag{21}$$

Furthermore, by the definition of $u(t,\beta)$ together with $s(t,\beta)$ defined as

(22) $s^{\Gamma}(t,\beta) = \phi^{\Gamma}(t) + \beta_h[\phi^{\Gamma}_{x^i}(t)\delta_h x^i(t) + \phi^{\Gamma}_{u^k}(t)\delta_h u^k(t)]$

$h = 1,\ldots,\bar{h} \quad \Gamma = 1,\ldots,K$

in (115-2) and (115-3) of [1], we obtain

(23) $$\frac{\partial u^k(t,0)}{\partial \beta_h} = U^k_{x^i}\frac{\partial x^i(t,0)}{\partial \beta_h} + U^k_{s^\Gamma}\left[\phi^{\Gamma}_{x^i}\delta_h x^i + \phi^{\Gamma}_{u^\tau}\delta_h u^\tau\right] \qquad \begin{matrix} k,\Gamma,\tau = 1,\ldots,K \\ h = 1,\ldots,\bar{h} \end{matrix}$$

where the bracketed term has argument t and represents the partial derivative of (22) with respect to β_h and where the argument of the partial derivatives of U are $(t,x(t,0),s(t,0))$.

Since $s^{\Gamma}(t,0) = \phi^{\Gamma}(t)$ and $x(t,0) = x_o(t)$ then by (102) of [1] we can write the relation (23) as

(24) $$\frac{\partial u^k(t,0)}{\partial \beta_h} = -\zeta^k_{\Gamma}\phi^{\Gamma}_{x^i}\frac{\partial x^i(t,0)}{\partial \beta_h} + \zeta^k_{\Gamma}\left[\phi^{\Gamma}_{x^i}\delta_h x^i + \phi^{\Gamma}_{u^\tau}\delta_h u^\tau\right] \qquad \begin{matrix} h = 1,\ldots,\bar{h} \\ \Gamma,k,\tau = 1,\ldots,K \end{matrix}$$

where the matrix $(\zeta_{\Gamma}^{\ k}(t))$ $k,\Gamma = 1,\ldots,K$ is the inverse of the matrix $(\phi^{\Gamma}_{u^k}(t))$ as introduced in (64) of [1]. Thus there results

(25) $$\frac{\partial u^k(t,0)}{\partial \beta_h} = \zeta^k_{\Gamma}\phi^{\Gamma}_{x^i}\left[\delta_h x^i - \frac{\partial x^i}{\partial \beta_h}(t,0)\right] + \delta_h u^\kappa \qquad h = 1,\ldots,\bar{h} \quad \Gamma,\kappa = 1,\ldots K$$

Now writing $x(t,\beta)$ for x in the right hand side of (119-1) of [1], then differentiating with respect to β_h at $\beta = 0$ and using (25) results in

(26) $$\frac{d}{dt}\left(\frac{\partial x^i(t,0)}{\partial \beta_h}\right) = f^i_{x^j}\frac{\partial x^j}{\partial \beta_h} + f^i_{u^k}\left[\zeta^k_{\Gamma}\phi^{\Gamma}_{x^\rho}(\delta_h x^\rho - \frac{\partial x^\rho}{\partial \beta_h}) + \delta_h u^k\right]$$

$k,\Gamma = 1,\ldots,K \qquad i,j,\rho = 1,\ldots,N \qquad h = 1,\ldots,\bar{h}$

where the unlisted arguments in the terms $\frac{\partial x}{\partial \beta}$ are $(t,0)$ and in the other terms is t. Then according to the definition of an admissible variation as listed above (8), together with (21), we see by substituting the terms $\delta_h x$ for $\frac{\partial x}{\partial \beta_h}(t,0)$ in (26) that these functions both satisfy the same set of differential

equations and pass through the same initial point. Then by the uniqueness of solutions to (26) and (21) we see that (13a) is proven. The relations (13b) and (13c) follow respectively from (13a) together with (25) and from the definition of the functions $\Theta^{\sigma}(\beta)$ of (121) of [1]. Thus Lemma 4.1 is proven.

5. Proof of the first two statements of Theorem 3.1

By the construction of Lemma 11.1 of [1] our family $a(\beta)$ of Lemma 4.1, for $\beta_h \geq 0$ $h = 1,\ldots\bar{h}$ is in the class A described above (71) of [1].

Now let $\tilde{\lambda}\rho$ ($\rho = 0,1,\ldots, p+2N$) be the constants of Theorem 8.2 of [1] and let $\delta_\omega a$ ($\omega = 1,\ldots,\Omega$) be the variations described in (7). Then by (70) in addition to Lemmas 11.1 and 15.1 and the inequality (155-2) all from [1] together with Lemma 3.1 and the definition of the terms $J'_\rho(a_o, \delta_\omega a)$ given in (8) we have that

$$(27) \qquad \tilde{\lambda}_\rho J'_\rho(a_o,\delta_\omega a) \geq 0 \qquad \rho = 0,1,\ldots,p+2N, \quad \omega = 1,\ldots,\Omega$$

where $J'_\rho(a_o,\delta_\omega a)$ is interpreted as a variation of J_ρ from the family $a(\beta)$ with $\beta_h \geq 0$ $(1 \leq h \leq \bar{h})$ of Lemma 4.1

Furthermore, according to our definition of an admissible variation, we see that the variations $\delta_\omega a$ $\omega = 1,\ldots,\Omega$ are also admissible. Thus by an argument analagous to that used above we see that (27) holds with $-\delta_\omega a$ replacing $\delta_\omega a$. Since

$$J'_\rho(a_o, -\delta_\omega a) = -J'_\rho(a_o,\delta_\omega a)$$

then we must have that

$$(28) \qquad \tilde{\lambda}_\rho J'_\rho(a_o,\delta_\omega a) = 0 \qquad \rho = 0,1,\ldots,p+2N; \qquad \omega = 1,\ldots,\Omega$$

(where Ω is always, the constant introduced above 10)).

Next, according to section 7 of [1], the multiplier $p_i(t)$, $\mu_\alpha(t)$, K^α of Theorems 6.1 and 8.2 of [1] are the terms of (76) of [1] and the relationship between the multipliers $\tilde{\lambda}$, λ respectively of Theorems 8.2 and 6.1 of [1] is by (92) and (76) of [1].

$$\lambda_\gamma = \tilde{\lambda}_\gamma \qquad \gamma = 0,1,\ldots,p \qquad \lambda_{p+i} = \tilde{\lambda}_{p+N+i} = K^\alpha \psi^\alpha_{x\,i}(t^o) \quad i = 1,\ldots,N$$

According to the definition of the multiplier $p_i(t)$ given in (76) and (66) of [1] we see that

$$p_i(t^1) = -\tilde{\lambda}_{p+i} \tag{29a}$$

and by the relationships indicated above (29) we see that with γ_k as the indices referred to in (9b)

$$0 = \lambda_{\gamma_k} = \tilde{\lambda}_{\gamma_k} \qquad k = 1,\ldots,\bar{p} \tag{29b}$$

Thus in (28) we may restrict ρ not to be any of the indices γ_k $k = 1,\ldots,\bar{p}$ and obtain

$$\tilde{\lambda}_\rho J'_\rho(a_o, \delta_\omega a) = 0 \qquad \rho = 0,1,\ldots,p+2N \qquad \rho \neq \gamma_k;\ \omega = 1,\ldots,\Omega \tag{30}$$

Now assume that $\tilde{\lambda}_o = 0$. Then by (29b) together with (30) and the non-singularity of the matrix of (10c) we must have

$$\tilde{\lambda}_\rho = 0 \qquad \rho = 0,1,\ldots,p+2N \tag{31}$$

According to the definition of $\mu_\alpha(t)$ (as given in (76) of [1]) and the relationships indicated above (29) and the non-singularity of the matrix of (108) of [1] we obtain

$$K^\alpha = 0$$
$$\mu_\alpha(t) = 0 \qquad t^o \le t \le t^1 \tag{32}$$

Thus by (31), (30) and (29a) there results in particular that

$$(32)\quad \tilde{\lambda}_\gamma = 0 \quad (0 \le \gamma \le p), \quad K^\alpha = 0, \quad p_i(t^1) = 0 \quad \mu_\alpha(t^1) = 0$$

which violates condition (81) of Theorem 8.2 of [1].

We have thus proven that if a_o is a normal solution then the multiplier $\tilde{\lambda}_o$ of Theorem 8.2 of [1] cannot vanish. In order to now prove the first statement of Theorem 3.1 we need only refer to the relationships above (29) to see that $\lambda_o = \tilde{\lambda}_o$. Thus the first statement of Theorem 3.1 is established.

In order to prove the second statement of Theorem 3.1 let z be the $\Omega + 1$ dimensional vector

$$z^o = J'_o(a_o, \delta a)$$

$$z^\ell = J'_{\gamma_\ell}(a_o, \delta a) \qquad 1 \le \gamma_\ell \le p;\ \gamma_\ell \ne \gamma_k;\ k = 1,\ldots,\bar{p}$$

$$z^{p-\bar{p}+j} = J'_{p+j}(a_o, \delta a) \qquad j = 1,\ldots,2N$$

for an admissible variation δa (where $\gamma_k, \bar{p}, \Omega$ are the terms introduced in (9) and $J'(a_o, \delta a)$ are the terms of (8)). In addition, let Λ be the $\Omega + 1$ dimensional vector with components $\tilde{\lambda}_\rho$ $\rho = 0,1,\ldots,p+2N$ $\rho \ne \gamma_k$. Then by (30) and the non-singularity of the matrix of (10c) we see that the vector z belongs to an Ω dimensional space Z satisfying

$$(33)\qquad (\Lambda, z) = 0 \qquad z \in Z$$

Thus Λ must belong to a one dimensional subset of vectors. Hence fixing $\tilde{\lambda}_o = 1$ fixes all components of the vector Λ and hence also all of $\tilde{\lambda}_\rho$ $\rho = 1,\ldots,p+2N$ $\rho \ne \gamma_k$. By using (29) and the definition of $p_i(t)$ and $\mu_\alpha(t)$ together with the relations indicated above (29) and the non-singularity of the matrix (108) from [1] we see that fixing $\tilde{\lambda}_\rho$ $\rho = 0,1,\ldots,p+2N$; $\rho \ne \gamma_k$ then also fixes $p_i(t)$, $\mu_\alpha(t)$, and K^α. Thus fixing $\tilde{\lambda}_o$ fixes all the $\tilde{\lambda}_\rho$, multipliers in addition to fixing $p_i(t)$, $\mu_\alpha(t)$, and K^α. The proof of the second statement of Theorem 3.1 then follows from the relation above (29) between the multipliers of Theorems 6.1 and 8.2 of [1]. Thus the second statement of Theorem 3.1 is established.

6. Proof of the Inequality (12)

As a next step, consider the admissible variations $\delta_\omega a \quad \omega = 1,\ldots,\Omega$ introduced in (10) together with an admissible variation y in the class Y introduced above (11). Thus we have the variations

$$\delta_\omega a: \quad \delta_\omega x(t) \quad \delta_\omega u(t) \quad \delta_\omega b. \qquad \omega = 1,\ldots,\Omega$$

$$y: \quad y(t) \quad v(t) \quad z \quad t^o \le t \le t^1$$

With the vector $\beta = (\beta_1,\ldots,\beta_\Omega)$ and π as parameters and referring to the proof of Lemma 11.1 of [1] we construct the family of arcs

$$a(\beta,\pi): \quad x(t,\beta,\pi) \quad u(t,\beta,\pi) \quad b(t,\beta,\pi) \quad |\beta| \le \bar{\beta}, \ |\pi| \le \bar{\pi}$$

(where $\bar{\beta}$ and $\bar{\pi}$ are certain positive constants) which satisfy the system of differential equations (119) of [1] but for the present case. Furthermore, an examination of the proof of that lemma will show that because of property (10b) of the variations $\delta_\omega a$, then the family so constructed satisfies

$$(34) \qquad \psi^\alpha(t,\beta,\pi) \le 0 \qquad |\beta| \le \bar{\beta} \qquad 0 \le \pi \le \bar{\pi} \qquad t^o \le t \le t^1$$

(i.e. with β not restricted to non-negative values). Thus for the last indicated ranges of β,π, the family $a(\beta,\pi)$ is in the class A defined above (71) of [1]. We shall henceforth understand the term, the family $a(\beta,\pi)$ to be that family with the parameter ranges of (34) unless otherwise specified.

As a next step, we see according to the definition of J_γ in (7a) that $J_\gamma(a_o) = I_\gamma(a_o) \quad \gamma = 0, 1,\ldots,p$. Thus with γ_k as the indices (9) we have

$$35) \qquad J_{\gamma_k}(a_o) < 0 \qquad k = 1,\ldots,\bar{p}.$$

Furthermore, according to the continuity properties of the functions g_γ and L_γ defining I_γ, there is a neighborhood 0 of the arc a_o such that a in 0 implies that $J_{\gamma_k}(a) < 0 \quad k = 1,\ldots,\bar{p}$.

By the above argument we then see that an arc a in the family $a(\beta,\pi)$ will

satisfy all the constraints of (1) if $\alpha \in 0$ and if

$$J_\gamma(\alpha(\beta,\pi)) \leq 0 \qquad \gamma \neq \gamma_k \qquad 1 \leq \gamma \leq p' \tag{36}$$

$$J_\rho(\alpha(\beta,\pi)) = 0 \qquad \rho = p' + 1,\ldots,p + 2N$$

We shall henceforth refer to an arc which satisfies all the conditions of (1) to be totally admissible.

In order to show that there is a one parameter family of totally admissible arcs which is contained in the family $\alpha(\beta,\pi)$, consider the system of equations

$$J_\rho(\alpha(\beta,\pi)) - \pi J'_\rho(\alpha_o, y) = 0 \qquad \rho = 1,\ldots,p + 2N \qquad \rho \neq \gamma_k \tag{37}$$

which is defined for $|\beta| \leq \bar{\beta}$, $|\pi| \leq \bar{\pi}$. According to Lemma 4.1, the functional determinant of (37) with respect to $\beta_1,\ldots\beta_\Omega$ at $\beta = \pi = 0$, is the matrix (10c). The system (37) has the initial solution point $(\beta,\pi) = (0,0)$. Then by the properties of the family $\alpha(\beta,\pi)$ for $|\beta| \leq \bar{\beta}$, $|\pi| \leq \bar{\pi}$ together with the Implicit Function Theorem and the non-singularity of the matrix of (10c), there is the solution to (37) as

$$\beta_\omega = \beta_\omega(\pi) \qquad \omega = 1,\ldots,\Omega, \quad |\pi| \leq \hat{\pi} \tag{38}$$

(where $\hat{\pi}$ is a suitable positive constant) of class C^2 and satisfying $\beta_\omega(0) = 0$.

By reducing if necessary the value of $\hat{\pi}$ we can guarantee that $\alpha(\beta(\pi),\pi) \in 0$ for $|\pi| \leq \hat{\pi}$ where 0 is the neighborhood introduced after (35).

Thus according to the properties of the variation y and the statement of (36) the family

$$\alpha(\beta(\pi),\pi) \qquad 0 \leq \pi \leq \hat{\pi} \tag{39}$$

(where $\hat{\pi}$ has been further reduced if necessary) is a totally admissible family of arcs with $\alpha(\beta(0),0) = \alpha_o$.

(1) Here, as always, γ_k are the indices of (9).

Furthermore, by substituting the solution $\beta_\omega(\pi)$ into (37) and differentiating with respect to π at $\pi = 0$, we obtain

$$(40)\quad J'_\rho(a_o,\delta_\omega a)\beta'_\omega(0) = 0 \qquad \rho = 1,\ldots,p+2N \qquad \rho \neq \gamma_k, \qquad \omega = 1,\ldots,\Omega$$

where $\beta'_\omega(0)$ denotes $\dfrac{d\beta_\omega(0)}{d\pi}$. By the non-singularity of the matrix of (10c) we must then have

$$(41)\qquad \beta'_\omega(0) = 0 \qquad \omega = 1,\ldots,\Omega\ .$$

Thus by Lemma 4.1 we obtain

$$(42)\quad \begin{aligned} \frac{dx(t,\beta(0),0)}{d\pi} &= \delta_\omega x(t)\beta'_\omega(0) + y(t) = y(t) \\ \frac{du(t,\beta(0),0)}{d\pi} &= \delta_\omega u(t)\beta'_\omega(0) + v(t) = v(t) \qquad t^o \le t \le t^1 \\ \frac{db(\beta(0),0)}{d\pi} &= \delta_\omega b\beta'_\omega(0) + z = z \end{aligned}$$

where the terms other than $\beta'_\omega(0)$ in the right hand side of (42) are those introduced above (34).

Now let G be the function of (76) of [1] and define the function J on the family $a(\beta(\pi),\pi)$ as

$$(43)\quad J(\pi) = [p_i(t^c)X^{ic}(b(\pi))]_{c=o}^{c=1} + \tilde{\lambda}_\gamma g_\gamma(b(\pi)) + \int_{t^o}^{t^1} G(s,\pi)ds +$$

$$\tilde{\lambda}_{p+N+i}[X^{io}(b(\pi)) - x^i(t^o,\pi)] \qquad \gamma = 0,1,\ldots,p$$

where: (i) $p(t)$ are the multipliers previously referred to (ii) X^{ic} and g_γ are defined in the problem statement and iii) the argument (π) means $(\beta(\pi),\pi)$ for the family of (39).

Differentiating (43) at $\pi = 0$ and using (42) yields

$$(44)\quad \frac{dJ(0)}{d\pi} = [p_i(t^c)\delta_\gamma X^{ic}]_{c=o}^{c=1} + \tilde{\lambda}_\gamma \delta_\gamma g_\gamma + \int_{t^o}^{t^1}[G_{x^i}y^i + G_{u^k}v^k]ds +$$

$$\tilde{\lambda}_{p+N+i}[\delta_\gamma X^{io} - y^i(t^o)]$$

where for example the term $\delta_\gamma X^{ic}$ is defined as $X^{ic}_{b^\sigma} z^\sigma$, i.e., the variation

in the functions X^{ic} due to the variation y and where in (44) we have used (42).

Furthermore by (80) and (89) of [1] we have

$$G_{x^i} = 0 \qquad G_{u^k} = 0$$

so that we may eliminate the integral in (44) and write that relation as

$$(46)\quad \frac{dJ(0)}{d\pi} = [p_i(t^c)\delta_y X^{ic}]_{c=o}^{c=1} + \tilde{\lambda}_\gamma \delta_y g_\gamma + \tilde{\lambda}_{p+N+i}[\delta_y X^{io} - y^i(t^o)] =$$

$$-\tilde{\lambda}_{p+N+i} y^i(t^o)$$

the last equality following from the transversatity condition (77) of [1].

According to the properties of our family of (37) as established in Lemma 4.1 together with the properties of the solution $\beta_\omega(\pi)$ of (41) we can differentiate (43) a second time to get (1)

$$(47)\quad \frac{d^2J(0)}{d\pi^2} = [p^i(t^c)(X^{ic}_{b^\alpha b^\tau} z^\sigma z^\tau + X^{ic}_{b^\sigma} \frac{d^2b^\sigma}{d\pi^2})]_{c=0}^{c=1} + \tilde{\lambda}_\gamma [g_{\gamma_{b^\sigma b^\tau}} z^\sigma z^\tau + g_{\gamma_{b^\sigma}} \frac{d^2b^\sigma}{d\pi^2}] +$$

$$\int_{t^o}^{t^1} [G_{x^i x^j} y^i y^j + 2G_{x^i u^h} y^i v^h + G_{u^h u^k} v^h v^k + G_{x^i} \frac{d^2x^i}{d\pi^2} + G_{u^h} \frac{d^2u^h}{d\pi^2}]ds +$$

$$\tilde{\lambda}_{p+N+i}[X^{io}_{b^\sigma b^\tau} z^\sigma z^\tau + X^{io}_{b^\sigma} \frac{d^2b^\sigma}{d\pi^2} - \frac{d^2x^i}{d\pi^2}(t^o)] \qquad \begin{matrix} i,j = 1,\ldots,N; & \sigma,\tau = 1,\ldots,r; \\ h,k = 1,\ldots,K; & \gamma = 0,1,\ldots,P \end{matrix}$$

where all derivatives with respect to b, x, u, are formed along a_o while all derivatives with respect to π are formed at $(\beta(\pi),\pi) = (\beta(0),0)$, i.e., at $\pi = 0$ on the family of (39).

As a next step we recall from the construction of Lemma 11.1 of [1] that the initial points $x(t^o,\beta(\pi),\pi)$ were selected with

$$x(t^o,\beta(\pi),\pi) = \bar{X}(\beta(\pi),\pi)$$

where the function $\bar{X}$ was constructed such that

(1) The terms $G_{x^i x^j}$ denote the second order partial derivative of G with respect to x^i, x^j and the other terms have similar definitions.

$$(48)\quad \psi^{\alpha}(t^{o},\bar{X}(\beta(\pi),\pi)) - [\psi^{\alpha}(t^{o}) + \psi^{\alpha}_{x^{i}}(t^{o})\ (\beta_{\omega}(\pi)\delta_{\omega}x^{i}(t^{o}) + \pi y^{i}(t^{o}))] = 0$$

$$\alpha = 1,\ldots,m \qquad \omega = 1,\ldots,\Omega$$

where the functions $\bar{X}$ are those of Lemma 10.4 of [1]. According to the properties of the functions involved, we can differentiate twice with respect to π at $\pi = 0$ to obtain

$$(49)\quad \psi^{\alpha}_{x^{i}}(t^{o})[\frac{d^{2}\bar{X}^{i}}{d\pi^{2}} - \beta''_{\omega}\delta_{\omega}x^{i}(t^{o})] + \psi^{\alpha}_{x^{i}x^{j}}(t^{o})\frac{d\bar{X}^{i}}{d\pi}\frac{d\bar{X}^{j}}{d\pi} = 0 \qquad \begin{matrix} i,j = 1,\ldots,N \\ \omega = 1,\ldots,\Omega \end{matrix}$$

where the unlisted arguments are at $\pi = 0$ and where $\beta''_{\omega}(0)$ denotes $\dfrac{d^{2}\beta_{\omega}(0)}{d\pi^{2}}$.

According to (77) from [1] together with (42) evaluated at t^{o}, (45) and the relations above (29) we may use (49) to re-write (47) as

$$(50)\quad \frac{d^{2}J(0)}{d\pi^{2}} = \Big[\Big[\sum_{c=o}^{c=1} p_{i}(t^{c})X^{ic}_{b^{\sigma}b^{\tau}}\Big] + \tilde{\lambda}_{\gamma}g_{\gamma}{}_{b^{\sigma}b^{\tau}}\Big]z^{\sigma}z^{\tau} + K^{\alpha}\psi^{\alpha}_{x^{i}}(t^{o})[X^{io}_{b^{\sigma}b^{\tau}}z^{\sigma}z^{\tau} -$$

$$\beta''_{\omega}(0)\delta_{\omega}x^{i}(t^{o})] + K^{\alpha}\psi^{\alpha}_{x^{i}x^{j}}(t^{o})y^{i}(t^{o})y^{j}(t^{o}) + \int_{t^{o}}^{t^{1}}[G_{x^{i}x^{j}}y^{i}y^{j} + 2G_{x^{i}u^{h}}y^{i}v^{h} +$$

$$G_{u^{h}u^{k}}v^{h}v^{k}]dt \qquad h,k = 1,\ldots,K; \qquad i,j = 1,\ldots,N;$$

$$\sigma,\tau = 1,\ldots,r; \qquad \gamma = 0,1,\ldots,P.$$

Recalling the definition of the function G of Theorem 6.1 of [1] as listed in (5) and using the relations indicated above (29) together with property (11f) for the variation y, then (50) may be written as

$$(51)\quad \frac{d^{2}J(0)}{d\pi^{2}} = \Big[\Big[\sum_{c=o}^{c=1} p_{i}(t^{c})X^{ic}_{b^{\sigma}b^{\tau}}\Big] + G_{b^{\sigma}b^{\tau}} + K^{\alpha}\psi^{\alpha}_{x^{i}x^{j}}(t^{o})X^{io}_{b^{\sigma}}X^{jo}_{b^{\tau}}\Big]z^{\sigma}z^{\tau}$$

$$-K^{\alpha}\psi^{\alpha}_{x^{i}}(t^{o})\beta''_{\omega}(0)\delta_{\omega}x^{i}(t^{o}) + \int_{t^{o}}^{t^{1}}[G_{x^{i}x^{j}}y^{i}y^{j} + 2G_{x^{i}u^{h}}y^{i}v^{h} + G_{u^{h}u^{k}}v^{h}v^{k}]dt$$

Referring to (43), we see that according to the definition of G in (76) of [1] and J_ρ $\rho = 0,1,\ldots,p + 2N$ in (8) there results

$$(52) \qquad J(\pi) = J_o(\pi) + \tilde{\lambda}_\rho J_\rho(\pi) + \Lambda(\pi) \qquad \rho = 1,\ldots,p + 2N$$

on the family of (39) where

$$\Lambda(\pi) \equiv \Lambda(\beta(\pi),\pi) = \int_{t^o}^{t^1} \mu_\alpha(s)\phi^\alpha(s,\beta(\pi),\pi)ds \qquad \alpha = 1,\ldots,m$$

and the functions μ_α, ϕ^α are as used in Section 1. In addition, according to the properties (11b) for the variation γ together with Lemma 2.1 and the property (10b) for the variations $\delta_\omega a$ we obtain with α not summed:

$$(53) \qquad \lim_{t\to\bar{t}} \left[\delta_\omega\psi^\alpha(\bar{t}) \frac{\mu_\alpha(t) - \mu_\alpha(\bar{t})}{t - \bar{t}}\right] = 0 \qquad \omega = 1,\ldots,\Omega$$

$$\lim_{t\to\bar{t}} \left[\delta_\gamma\psi^\alpha(\bar{t}) \frac{\mu_\alpha(t) - \mu_\alpha(\bar{t})}{t - \bar{t}}\right] = 0 \qquad t^o \le \bar{t} \le t^1. \qquad \alpha \text{ not summed.}$$

Thus according to the admissibility of[1] the variations $\delta_\omega a$, γ and the continuity properties of μ_α as expressed in Theorem 6.1 of [1], we have

$$(54a) \quad \frac{d}{dt}[\mu_\alpha(\bar{t})\delta_\omega\psi^\alpha(\bar{t})] = \lim_{t\to\bar{t}} \frac{\mu_\alpha(t)\delta_\omega\psi^\alpha(t) - \mu_\alpha(\bar{t})\delta_\omega\psi^\alpha(\bar{t})}{t - \bar{t}} =$$

$$\lim_{t\to\bar{t}} \mu_\alpha(t) \frac{\delta_\omega\psi^\alpha(t) - \delta_\omega\psi^\alpha(\bar{t})}{t - \bar{t}} + \lim_{t\to\bar{t}} \delta_\omega\psi^\alpha(\bar{t}) \frac{\mu_\alpha(t) - \mu_\alpha(\bar{t})}{t - \bar{t}} =$$

$$\mu_\alpha(\bar{t})\delta_\omega\phi^\alpha(\bar{t}) \qquad \omega = 1,\ldots,\Omega \qquad \alpha \text{ not summed.}$$

(1) See 11.1) of [1].

and similarly for γ so that

$$(54b)\quad \frac{d}{dt}\,[\mu_\alpha(\bar{t})\delta_\gamma\psi^\alpha(\bar{t})] = \mu_\alpha(\bar{t})\delta_\gamma\phi^\alpha(\bar{t}) \qquad \alpha = 1,\ldots,m \qquad t^o \le t \le t^1 .$$

α not summed.

In order now to evaluate the term $\Lambda(\pi)$ introduced above, we recall from the construction of Lemma 11.1 of [1] but for the present case that

$$(55)\quad \phi^\alpha(t,\beta(\pi),\pi) = \phi^\alpha(t) + \beta_\omega(\pi)\delta_\omega\phi^\alpha(t) + \pi\delta_\gamma\phi^\alpha(t) \qquad \alpha = 1,\ldots,m; \qquad t^o \le t \le t^1.$$

Then as the family $a(\beta(\pi),\pi)$ contains a_o for $\pi = 0$ we must have

$$(56)\quad \Lambda(\pi) - \Lambda(0) = \int_{t^o}^{t^1} \mu_\alpha(s)[\beta_\omega(\pi)\delta_\omega\phi^\alpha(s) + \pi\delta_\gamma\phi^\alpha(s)]ds.$$

However, by (54) this becomes

$$(57)\quad \Lambda(\pi) - \Lambda(0) = \int_{t^o}^{t^1} \pi[\frac{d}{ds}[\mu_\alpha(s)\delta_\gamma\psi^\alpha(s)] + \beta_\omega(\pi)\frac{d}{ds}[\mu_\alpha(s)\delta_\omega\psi^\alpha(s)]]ds =$$

$$[\mu_\alpha(t^c)(\beta_\omega(\pi)\delta_\omega\psi^\alpha(t^c) + \pi\delta_\gamma\psi^\alpha(t^c))]_{c=o}^{c=1} = 0$$

the last equality following from properties (11a) of the variation γ and from properties (10a) and (10b) of the variations $\delta_\omega a$ together with the property listed below (6) for the multipliers $\mu_\alpha(t)$. Thus by (52) we have

$$(58)\quad J(\pi) - J(0) = J_o(\pi) - J_o(0) + \tilde{\lambda}_\rho[J_\rho(\pi) - J_\rho(0)] \qquad \rho = 0,1,\ldots,p + 2N.$$

Next, by (29), together with the definition of the family of (39) under consideration, the relation (37) and the properties of the variation γ there results

$$(59)\quad \tilde{\lambda}_\rho J_\rho(\pi) = 0 \qquad \rho = 1,\ldots,p + 2N \qquad \rho \text{ not summed.}$$

Then (58) becomes

$$J(\pi) - J(0) = J_o(\pi) - J_o(0) \tag{60}$$

for the family of (39).

However since the arc a_o is a solution to our problem and as the family of (39) is totally admissible, then we must have that

$$J(\pi) - J(0) \geq 0 \tag{61}$$

Now according to the relation between the function H of Theorem 6.1 of [1] and the function G of (51)

$$H = -\dot{p}_i(t)x^i - G \tag{62}$$

as given in (93) of [1], we see that in (51) the derivatives of G may be replaced by the derivatives of H. Then by (46),(51) and these remarks, the inequality (12) will be proven if we prove that

$$\tilde{\lambda}_{p+N+i} y^i(t^o) = 0 \tag{63a}$$

$$\tilde{\lambda}_{p+N+i} \beta''_\omega(0)\delta_\omega x^i(t^o) = K^\alpha \psi^\alpha_{x^i}(t^o)\beta''_\omega(0)\delta_\omega x^i(t^o) = 0 \qquad \omega = 1,\ldots,\Omega \tag{63b}$$

the first equality in (63b) following from the relations above (29). This result will be established in the following section.

7. An Auxiliary Lemma.

Lemma 7.1 Let α_j $j = 1,\ldots,\hat{m}$ be those indices such that $\psi^\alpha(t^o) = 0$. Then given the N dimensional vector $D = (d^1,\ldots,d^N)$ satisfying

(64) $$\psi^{\alpha}_{x^i}{}^{j}(t^o)d^i = 0 \qquad j = 1,\ldots,\hat{m}$$

$$\psi^{\beta}_{x^i}(t^o)d^i = 0 \qquad \text{if} \quad \mu_\beta(t^o) \neq 0$$

the relation

(65) $$\tilde{\lambda}_{p+n+i}d^i = 0$$

is also true.

Proof: According to the above definitions together with the properties of the functions ψ^α, we may select a positive constant δ so small that

(66) $$\psi^\beta(t) < 0 \qquad \beta \neq \alpha_j \qquad j = 1,\ldots,\hat{m} \qquad t^o \leq t \leq t^o + \delta$$

Define the K dimensional arc w satisfying

(67a) $$w^\alpha(t^o) = \psi^\alpha_{x^i}(t^o)d^i$$

(67b) $$\dot{w}^\alpha(t^o) = \begin{cases} (-\psi^\alpha_{x^i}(t^o)d^i)\frac{2}{\delta} & t^o \leq t \leq t^o + \delta/2 \\ 0 & t^o + \delta/2 \leq t \leq t^1 \end{cases} \qquad \alpha = 1,\ldots,m$$

(67c) $$w^\Gamma(t) \equiv 0 \qquad \Gamma = m+1,\ldots,K \qquad t^o \leq t \leq t^1$$

Then w is in the class W of Section 13 of [1]. By Lemma 13.1 of [1], we can find an admissible variation

$$\delta a: \qquad \delta x(t) \qquad \delta u(t) \qquad \delta b \qquad t^o \leq t \leq t^1$$

satisfying

(68a) $\delta x^{j_s}(t^o) = d^{j_s} \qquad j_s \neq i_\rho \qquad s = 1,\ldots,N - m$

(68b) $\delta b = 0$

where i_ρ are the indices of (108) of [1] and such that

(68c) $\psi^\alpha_{x^i}(t)\delta x^i(t) \equiv \delta\psi^\alpha(t) = w^\alpha(t) \qquad \alpha = 1,\ldots,m$

(68d) $\delta\phi^\Gamma(t) = w^\Gamma(t) \qquad \Gamma = m + 1,\ldots,K \qquad t^o \leq t \leq t^1$

According to the above, and due to the admissibility of δa we have that

(69a) $$\delta\phi^\alpha(t) = \frac{d}{dt}\delta\psi^\alpha(t) = \dot{w}^\alpha(t) = \begin{cases} (-\psi^\alpha_{x^i}(t^o)d^i)\frac{2}{\delta} & t^o \leq t \leq t^o + \delta/2 \\ 0 & t^o + \delta/2 \leq t \leq t^1 \end{cases}$$

and by (67c) and (68d) also

(69b) $\delta\phi^\Gamma(t) = 0 \qquad \Gamma = m + 1,\ldots,K \qquad t^o \leq t \leq t^1$

Furthermore, by (67a) and (68a) together with (68c) evaluated at t^o we obtain

(70) $\psi^\alpha_{x^{i_\rho}}(t^o)[d^{i_\rho} - \delta x^{i_\rho}(t^o)] = 0 \qquad \rho,\alpha = 1,\ldots,m.$

where i_ρ are the indices of (68a). Then by the non-singularity of the matrix $(\psi^\alpha_{x^{i_\rho}}(t^o))$ (see 108 of [1]) we see that $\delta x^{i_\rho}(t^o) = d^{i_\rho} \quad \rho = 1,\ldots,m$ so that together with (68a) this becomes

(71) $\delta x^i(t^o) = d^i \qquad i = 1,\ldots,N$

Next, by 155-2) and Lemmas 11.1 and 15.1 all of [1] together with (69) and (68b) we get by computing the variations of our functionals J_ρ with the above formed variation, that

(72) $$\tilde{\lambda}_\rho \int_{t^o}^{t^o + \delta/2} F_{\rho u} {}^k \zeta_\alpha {}^k (-\psi^\alpha_{x^i}(t^o)d^i)\frac{2}{\delta}\ dt - \tilde{\lambda}_{p+N+i} d^i \geq 0 \qquad \rho = 0,1,\ldots,p+$$

(where the functions F_ρ and $\zeta_\alpha {}^k$ are defined respectively in (67) and (64) of [1]). Recalling the definition of the multipliers $\mu_\alpha(t)$ given in (74) and (76) both of [1], we see that (72) may be written in the form

(73) $$(-\psi^\alpha_{x^i}(t^o)d^i)\frac{2}{\delta}\int_{t^o}^{t^o + \delta/2} \mu_\alpha dt - \tilde{\lambda}_{p+N+i} d^i \geq 0.$$

Now if $\alpha = \alpha_j$ $1 \leq j \leq \hat{m}$ (where these are the indices of (64)) then the first term of (73) vanishes, while if $\alpha \neq \alpha_j$ $j = 1,\ldots,\hat{m}$ then by Lemma 2.1 we know that $\mu_\alpha(t)$ is constant on $[t^o, t^o + \delta/2]$ so that the first term of (73) becomes for each such α

(74) $$-\mu_\alpha(t^o)\psi^\alpha_{x^i}(t^o)d^i \qquad \alpha \text{ not summed.}$$

which according to the second part of (64) also vanishes. Thus the first term of (73) vanishes for $\alpha = 1,\ldots,m$ and we get

(75) $$-\tilde{\lambda}_{p+N+i} d^i \geq 0.$$

Now replace the vector D of the bypothesis, by $-D$. The above argument holds in an analgous fashion for $-D$ with the analogue of (75) being

$$-\tilde{\lambda}_{p+N+i}(-d^i) \geq 0 \tag{76}$$

Thus we obtain

$$\tilde{\lambda}_{p+N+i} d^i = 0. \tag{77}$$

and the lemma is proven.

By referring to properties (10a) and (10b) for the variations $\delta_\omega a$ and to properties (11a) and (11c) for the variation y, we see that the hypotheses of Lemma 7.1 is satisfied by these variations. Thus (63) is proven and Theorem 3.1 is established.

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A set of second order necessary conditions is obtained for solution to a basic problem in optimal control involving the state constraints

$$\psi^{\alpha}(t,x) \leq 0 \qquad \alpha = 1,\ldots,m.$$

For the case of a normal solution arc, the methods of Hestenes are used to show that the second variation of a certain functional is non-negative on a class of variations of the solution arc. The results for this problem are basic in the sense that they can be extended to include analagous results for much more complicated problems.

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